Assignment 7

This homework is due *Thursday* Oct 22.

There are total 28 points in this assignment. 25 points is considered 100%. If you go over 25 points, you will get over 100% for this homework and it will count towards your course grade (not over 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.5, 3.6 in Bartle–Sherbert.

1. Cauchy sequences

(1) [2pt] (Exercise 3.5.4) Show directly from definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.

(2) [3pt] (3.5.9) If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence. (*Hint:* $x_{n+2} - x_n = (x_{n+2} - x_{n+1}) + (x_{n+1} - x_n)$. Generalize this to $x_{n+m} - x_n$.)

(3) [2pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$. Is it true that if for any $\varepsilon > 0$, there is a natural number $H = H(\varepsilon)$ such that for all $n > H$, $|x_n - x_{n+1}| < \varepsilon$, then X is a Cauchy sequence? (*Hint*: inspect partial sums of harmonic series[1], or look at $x_n = \sqrt{n}$).

(4) [2pt] Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that x_n is ultimately constant.

2. Properly divergent sequences

In the exercises below we look at sequences that "go to infinity".

Definition. Let (x_n) be a sequence of real numbers. We say that (x_n) *tends to* (diverges to) $+\infty$, and write $\lim(x_n) = +\infty$, if for every $\alpha \in \mathbb{R}$ there exists a natural number K such that if $n > K$, then $x_n > \alpha$.

Another notation is $x_n \to +\infty$ $(n \to \infty)$.

(4) [2pt] Give an analogous definition of a sequence that tends to $-\infty$.

Definition. We say that (x_n) is *properly divergent* in case we have either $\lim(x_n) = +\infty$ or $\lim(x_n) = -\infty$.

(5) ($\sim$ Example 3.6.2) For the following sequences determine whether they are properly divergent.
 (a) [1pt] $x_n = n^2$.
 (b) [1pt] $x_n = (-1)^n n$.
 (c) [2pt] $x_n = c^n$, where c is a given real number. (*Hint:* Note that the answer depends on c. For $c > 1$, use Bernoulli's inequality.)

— see next page —

[1] If you don't know what those are, wait until Tuesday or follow other suggestion in the hint.

(6) [2pt] (Theorem 3.6.4) Prove that following theorem (we may call it the squeeze theorem for properly divergent sequences).
Let (x_n) and (y_n) be two sequences of real numbers and suppose that

$$x_n \leq y_n \text{ for all } n \in \mathbb{N}.$$

If $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.
If $\lim(y_n) = -\infty$, then $\lim(x_n) = -\infty$.

(7) [3pt] Give examples of sequences (x_n) and (y_n) that tend to $+\infty$ and:
(a) $\lim(x_n/y_n) = 0$,
(b) $\lim(x_n/y_n) = 10$,
(c) $\lim(x_n/y_n) = +\infty$,
(d) $\lim(x_n/y_n)$ does not exist as either a real number or infinity.

(8) [3pt] (Theorem 3.6.5) Prove that following theorem.
Let (x_n) and (y_n) be two sequences of positive real numbers and suppose that for some $L \in \mathbb{R}$, $L > 0$, we have

$$\lim(x_n/y_n) = L.$$

Then $\lim(x_n) = +\infty$ if and only if $\lim(y_n) = +\infty$.
(*Hint:* Show that for n large enough, $\frac{1}{2}L < x_n/y_n < \frac{3}{2}L$. Then use the previous theorem.)

(9) ($\sim$ Exercise 3.6.8, 10) For the following sequences determine whether they are properly divergent.
(a) [2pt] $x_n = \sqrt{n^2 - 1}/\sqrt{n + 100}$. (*Hint:* Use the theorem above.)
(b) [1pt] $x_n = \sin\sqrt{n}$.
(c) [2pt] x_n if it is given that $\lim(x_n/n) = L$, where $L > 0$. (*Hint:* Don't forget that to use the previous theorem, you have to explain why $x_n > 0$ first.)